ISRAEL JOURNAL OF MATHEMATICS 84 (1993), 229–238

A QUESTION IN THE THEORY OF SATURATED FORMATIONS OF FINITE SOLUBLE GROUPS

BY

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ABSTRACT

This paper examines the following question. If $\mathcal{H}$ and $\mathcal{F}$ are saturated formations then $\mathcal{H}_{\mathcal{F}}$ is defined to be the class of all soluble groups whose $\mathcal{H}$-normalizers belong to $\mathcal{F}$. In general $\mathcal{H}_{\mathcal{F}}$ is a formation, but need not be a saturated formation. Here the smallest saturated formation containing $\mathcal{H}_{\mathcal{F}}$ is studied.

Introduction. Preliminaries

All groups considered in this paper are finite and soluble. The reader is assumed to be familiar with the theory of saturated formations of finite soluble groups. We shall adhere to the notation used in [6]: this book is the main reference for the basic notation, terminology and results.

Received January 10, 1993 and in revised form March 1, 1993

First let us introduce the question we aim to analyze in this paper together with the specific notation which is used throughout.

Let $\mathcal{H}$ be a saturated formation, $\mathcal{H} = LF(H)$, where H is the canonical local definition of $\mathcal{H}$, and $\mathcal{F}$ a formation. Let us denote by $\mathcal{H}_{\mathcal{F}}$ the class of all groups whose $\mathcal{H}$-normalizers belong to $\mathcal{F}$. Since $\mathcal{H}_{\mathcal{F}} = \mathcal{H}_{\mathcal{H}\cap\mathcal{F}}$ it can be supposed in the sequel that $\mathcal{F} \subseteq \mathcal{H}$. If $\mathcal{F}$ is also saturated we will denote $\mathcal{F} = LF(F)$, where F is the canonical local definition of $\mathcal{F}$. This means that $F(p) \subseteq H(p)$ for all $p \in \operatorname{char}(\mathcal{F})$ (see [6], Prop. IV.3.11.(c)).

The class $\mathcal{X} = \mathcal{H}_{\mathcal{F}}$ is a formation but in general $\mathcal{X}$ is not saturated. Our purpose is to study its saturation $\overline{\mathcal{X}}$, i.e. $\overline{\mathcal{X}} =< \mathrm{Q}, \mathrm{R}_0, \mathrm{E}_\Phi > \mathcal{X}$, the smallest saturated formation containing $\mathcal{X}$; the canonical local definition of $\overline{\mathcal{X}}$ is denoted by $\overline{X}$: $\overline{\mathcal{X}} = LF(\overline{X})$.

The analogous question for projectors, i.e. the saturation of the formation $\mathcal{H} \downarrow \mathcal{F} = (G\colon \operatorname{Proj}_{\mathcal{H}}(G) \subseteq \mathcal{F})$, was fully analyzed by Doerk in [1] (see [6] pp. 503–508).

Formations of the type $\mathcal{H}_{\mathcal{F}}$ appear in the maximal local definition of a saturated formation:

THEOREM (see [6] Th. V.3.18): *If F is the canonical local definition of the saturated formation $\mathcal{F}$ then the formation function $f(p) = \mathcal{F}_{F(p)}$ is the largest local definition of $\mathcal{F}$, i.e. if h is a formation function such that $\mathcal{F} = LF(h)$ then $h(p) \subseteq f(p)$ for all primes p.*

There exists a trivial case of saturation of the class $\mathcal{H}_{\mathcal{F}}$: if $\mathcal{F} = \mathcal{H}$ then $\mathcal{H}_{\mathcal{H}} = \mathcal{S}$, the class of all (soluble) groups. So, we assume throughout the paper that $\mathcal{F}$ is a proper saturated subformation of $\mathcal{H}$.

A description of the class $\mathcal{H}_{\mathcal{F}}$

PROPOSITION 1:

(a) *Given a group G the following statements are pairwise equivalent:*

  (i) $G \in \mathcal{H}_{\mathcal{F}}$,

  (ii) $\operatorname{Nor}_{\mathcal{H}}(G) = \operatorname{Nor}_{\mathcal{F}}(G)$,

  (iii) *all $\mathcal{H}$-central chief factors of G are $\mathcal{F}$-central,*

(b) $\mathcal{H}_{\mathcal{F}} \subseteq \mathcal{K} = (G\colon G^{\mathcal{H}} = G^{\mathcal{F}})$.

(c) $\mathcal{H}_{\mathcal{F}} = \mathcal{K}$ *if and only if $\mathcal{H}_{\mathcal{F}}$ is a saturated formation.*

Proof: (a) It is clear by the cover and avoidance properties of the normalizers (see [6] Th. V.3.2) that $\mathcal{H}_{\mathcal{F}}$ is the class of all groups such that all $\mathcal{H}$-central chief factors are $\mathcal{F}$-central. Notice that in every group, $\mathcal{H}$-critical maximal subgroups are indeed $\mathcal{F}$-critical maximal subgroups and then every $\mathcal{H}$-normalizer contains an $\mathcal{F}$-normalizer. So, if $G \in \mathcal{H}_{\mathcal{F}}$ we have $\mathrm{Nor}_{\mathcal{H}}(G) = \mathrm{Nor}_{\mathcal{F}}(G)$.

(b) Let $G \in \mathcal{H}_{\mathcal{F}}$; if $D \in \mathrm{Nor}_{\mathcal{H}}(G)$ then $D \in \mathcal{F}$ and $G = DG^{\mathcal{H}}$. Then $G/G^{\mathcal{H}} \in \mathcal{F}$ and $G^{\mathcal{H}} = G^{\mathcal{F}}$. So, $\mathcal{H}_{\mathcal{F}} \subseteq \mathcal{K}$.

(c) Suppose that $\mathcal{H}_{\mathcal{F}}$ is saturated and let G be a group in $\mathcal{K} \setminus \mathcal{H}_{\mathcal{F}}$ of minimal order. Since $\mathcal{K}$ is a homomorph the group G is in $b(\mathcal{H}_{\mathcal{F}})$ and then G is a primitive group; denote by N its minimal normal subgroup and suppose that N is a p-group; since $G/N \in \mathcal{H}_{\mathcal{F}}$, if $D \in \mathrm{Nor}_{\mathcal{H}}(G)$ then $D/(D \cap N) \cong DN/N \in \mathcal{F}$ and therefore $N \leq D$ which means that N is $\mathcal{H}$-central. Recall that $N = C_G(N)$. So, $G/N \in H(p)$ and then $G \in \mathcal{S}_p H(p) = H(p) \subseteq \mathcal{H}$. This implies that $G^{\mathcal{H}} = 1$ and since $G \in \mathcal{K}$ we have $G^{\mathcal{F}} = G^{\mathcal{H}} = 1$ and then $G \in \mathcal{F}$. But this implies that $G \in \mathcal{H}_{\mathcal{F}}$, a contradiction. Hence $\mathcal{H}_{\mathcal{F}} = \mathcal{K}$.

Conversely, suppose $\mathcal{H}_{\mathcal{F}} = \mathcal{K}$ and suppose that there exists a group G of minimal order in $\mathrm{E}_{\Phi}\mathcal{K} \setminus \mathcal{K}$. The group G is monolithic and its socle N is a minimal normal subgroup of G contained in $\Phi(G)$. Since $G/N \in \mathcal{K}$ we have $G^{\mathcal{F}}N = G^{\mathcal{H}}N$ and since $G^{\mathcal{H}} \leq G^{\mathcal{F}}$ we have $G^{\mathcal{F}} = G^{\mathcal{H}}(G^{\mathcal{F}} \cap N)$. Since $G \notin \mathcal{K}$ we have $N \leq G^{\mathcal{F}}$ and $G^{\mathcal{F}} = G^{\mathcal{H}}N$. But $G^{\mathcal{F}}/G^{\mathcal{H}} = NG^{\mathcal{H}}/G^{\mathcal{H}} \leq \Phi(G)G^{\mathcal{H}}/G^{\mathcal{H}} \leq \Phi(G/G^{\mathcal{H}})$. Since $G/G^{\mathcal{F}} \in \mathcal{F}$ and $\mathcal{F}$ is saturated we have $G/G^{\mathcal{H}} \in \mathcal{F}$ and then $G \in \mathcal{K}$, a contradiction. Hence $\mathcal{K}$ is a saturated formation. ∎

The canonical local definition of the saturation of $\mathcal{H}_{\mathcal{F}}$

LEMMA 1:

(i) *We have always* $\mathcal{H}_{F(p)} \subseteq \overline{X}(p) \subseteq \mathcal{S}_p\mathcal{X}$.

(ii) *If moreover* $\mathcal{F} \not\subseteq H(p)$, *for some prime* p, *then* $\overline{X}(p) = \mathcal{S}_p\mathcal{X}$. *Hence, if the formation* $\mathcal{X}$ *is saturated, i.e.* $\mathcal{X} = \overline{\mathcal{X}}$, *and* $\mathcal{F} \not\subseteq H(p)$, *for some prime* p, *then* $\mathcal{S}_p\mathcal{X} = \mathcal{X}$.

Proof: (i) The class $\mathcal{N}\mathcal{X}$ is a saturated formation. So $\overline{\mathcal{X}} \subseteq \mathcal{N}\mathcal{X}$ and then $\overline{X}(p) \subseteq \mathcal{S}_p\mathcal{X}$ for every prime number p, (see [6], Prop. IV.3.11.(c)).

Suppose there exists a group $G \in \mathcal{H}_{F(p)} \setminus \overline{X}(p)$ of minimal order. Then G is monolithic and its socle is a minimal normal p'-subgroup. Let V be an irreducible and faithful G-module over $\mathrm{GF}(p)$ and construct the semidirect product $Y = [V]G$, (see [6], Cor. B.10.7) for the existence of such a module). If $Y \in \mathcal{H}$ then $G \in \mathcal{H}$ and, since $G \in \mathcal{H}_{F(p)}$, we have $G \in F(p)$; but hence $Y \in \mathcal{S}_p F(p) \subseteq F(p) \subseteq \mathcal{F}$ and $Y \in \mathcal{X} \subseteq \overline{\mathcal{X}}$; now $G \cong Y/O_{p'p}(Y) \in \overline{X}(p)$, a contradiction. So, $Y \notin \mathcal{H}$ and $\mathrm{Nor}_{\mathcal{H}}(G) \subseteq \mathrm{Nor}_{\mathcal{H}}(Y)$, by [6] Prop. V.3.8, and then $Y \in \mathcal{X} \subseteq \overline{\mathcal{X}}$ and again $G \in \overline{X}(p)$, a contradiction. Therefore $\mathcal{H}_{F(p)} \subseteq \overline{X}(p)$.

(ii) Suppose that $\mathcal{F} \not\subseteq H(p)$ for some prime p and $\overline{X}(p) \neq \mathcal{S}_p\mathcal{X}$; let G be a group of minimal order in $\mathcal{S}_p\mathcal{X} \setminus \overline{X}(p)$. Then G is a monolithic group whose socle is a minimal normal p'-subgroup of G. Hence, in fact, $G \in \mathcal{X}$. Let V be an irreducible and faithful G-module over $\mathrm{GF}(p)$ and construct the semidirect product $Y = [V]G$.

If $Y \notin \mathcal{H}$ then $\mathrm{Nor}_{\mathcal{H}}(G) \subseteq \mathrm{Nor}_{\mathcal{H}}(Y)$ and then $Y \in \mathcal{X} \subseteq \overline{\mathcal{X}}$ and then $Y/O_{p'p}(Y) \cong G \in \overline{X}(p)$, a contradiction. Therefore $Y \in \mathcal{H}$ and $G \in H(p) \subseteq \mathcal{H}$. So, $G \in \mathcal{X} \cap \mathcal{H} = \mathcal{F}$.

Let H be a group of minimal order in $\mathcal{F} \setminus H(p)$. Again H is a monolithic group whose socle is a minimal normal p'-subgroup. Construct the direct product $D = G \times H$; then $D \in \mathcal{F}$ and by [6] Cor. B.10.7 there exists an irreducible and faithful D-module W over $\mathrm{GF}(p)$. Construct the semidirect product $K = [W]D$. If $K \notin \mathcal{H}$ then $D \in \mathrm{Nor}_{\mathcal{H}}(K)$ and then $K \in \mathcal{X} \subseteq \overline{\mathcal{X}}$ and $G \cong K/O_{p'p}(K) \in \overline{X}(p)$, a contradiction. Hence $W \in \mathcal{H}$ and $D \in H(p)$; but this implies that $H \in H(p)$ a contradiction.

So, our claim holds and $\overline{X}(p) = \mathcal{S}_p\mathcal{X}$.

Finally, notice that if $\mathcal{X}$ is saturated and $\mathcal{F} \not\subseteq H(p)$ then $\overline{X}(p) = \mathcal{S}_p\mathcal{X} \subseteq \overline{\mathcal{X}} = \mathcal{X}$. ∎

LEMMA 2: *Assume that $\mathcal{X} = \mathcal{H}_{\mathcal{F}}$ is a saturated formation, i.e. $\mathcal{X} = \overline{\mathcal{X}}$. If $\mathcal{F} \subseteq H(p)$ then $\overline{X}(p) = \mathcal{H}_{F(p)}$.*

Hence, if the formation $\mathcal{X} = \mathcal{H}_{\mathcal{F}}$ is saturated then

$$\overline{X}(p) = \begin{cases} \mathcal{H}_{F(p)}, & \text{if } \mathcal{F} \subseteq H(p); \\ \mathcal{S}_p\mathcal{X}, & \text{if } \mathcal{F} \not\subseteq H(p). \end{cases}$$

Proof: By Lemma 1(i) we have $\mathcal{H}_{F(p)} \subseteq \overline{X}(p)$. Let G be a group of minimal order in $\overline{X}(p) \setminus \mathcal{H}_{F(p)}$. Then G is a monolithic group. Suppose that $N = \mathrm{Soc}(G)$ is a p-group. Then $G \in \mathcal{S}_p\mathcal{H}_{F(p)}$. Let $D \in \mathrm{Nor}_{\mathcal{H}}(G)$; then $DN/N \in \mathrm{Nor}_{\mathcal{H}}(G/N) \subseteq F(p)$ and then $D \in \mathcal{S}_pF(p) = F(p)$. Therefore we have indeed $G \in \mathcal{H}_{F(p)}$, a contradiction.

Hence, $\mathrm{Soc}(G)$ is a minimal normal p'-subgroup and by Lemma 1(i) we have $G \in \mathcal{X}$.

If $G \notin \mathcal{H}$ and $D \in \mathrm{Nor}_{\mathcal{H}}(G)$ then D is a proper subgroup of G and since D is a well-placed subgroup of G we have $D \in \overline{X}(p)$ by [6] Cor. IV.1.15(a). Minimality of G forces $D \in \mathcal{H}_{F(p)}$ and this implies $G \in \mathcal{H}_{F(p)}$, a contradiction.

So, $G \in \mathcal{H}$; then $G \in \mathcal{X} \cap \mathcal{H} = \mathcal{F} \subseteq H(p)$. Let V be an irreducible and faithful G-module over GF(p) and construct the semidirect product $Y = [V]G$. Then $Y \in \mathcal{S}_p H(p) = H(p) \subseteq \mathcal{H}$. On the other hand $Y \in \mathcal{S}_p \overline{X}(p) = \overline{X}(p) \subseteq \overline{\mathcal{X}} = \mathcal{X}$ and therefore $Y \in \mathcal{H} \cap \mathcal{X} = \mathcal{F}$; so $G \cong Y/O_{p'p}(Y) \in F(p)$ and the $G \in \mathcal{H}_{F(p)}$, final contradiction.

Therefore our claim is true. ∎

Proposition 2: *The formation $\mathcal{X} = \mathcal{H}_{\mathcal{F}}$ is saturated, i.e. $\mathcal{X} = \overline{\mathcal{X}}$, if and only if the following two conditions hold:*

(i) *if $\mathcal{F} \subseteq H(p)$ then $\overline{X}(p) = \mathcal{H}_{F(p)}$, and*

(ii) *if $\mathcal{F} \not\subseteq H(p)$ then $H(p) \cap \mathcal{F} = F(p)$.*

Proof: Suppose that $\mathcal{X}$ is saturated. Then part (i) holds by Lemma 2. If $\mathcal{F} \not\subseteq H(p)$ then $\overline{X}(p) = \mathcal{S}_p\mathcal{X}$ by Lemma 1(ii). Consider a group G of minimal order in $(H(p) \cap \mathcal{F}) \setminus F(p)$. As usual, G is a monolithic group whose socle is a minimal normal p'-subgroup. If V is an irreducible and faithful G-module over GF(p), we can construct the semidirect product $Y = [V]G$. Now $Y \in \mathcal{S}_p H(p) = H(p) \subseteq \mathcal{H}$. On the other hand $G \in \mathcal{X}$. By Lemma 1, $Y \in \mathcal{X}$ and then $Y \in \mathcal{H} \cap \mathcal{X} = \mathcal{F}$. Hence $G \cong Y/O_{p'p}(Y) \in F(p)$, a contradiction. So, (ii) holds.

Conversely, suppose that (i) and (ii) hold and let G be a group of minimal order in $\overline{\mathcal{X}} \setminus \mathcal{X}$. The group G is monolithic and its socle is a minimal normal p-subgroup for some prime p; so, $O_{p'}(G) = 1$. Notice that $G \in \overline{X}(p)$.

If $\mathcal{F} \not\subseteq H(p)$ then, by Lemma 1(ii), $G \in \overline{X}(p) = \mathcal{S}_p\mathcal{X}$. If $G \notin \mathcal{H}$ and $D \in \mathrm{Nor}_{\mathcal{H}}(G)$ then D is a proper subgroup of G. Since D is a well-placed subgroup of G, $D \in \overline{\mathcal{X}}$ by [6] Cor. IV.1.15(a) again; by minimality of G we really have $D \in \mathcal{X}$. Thus, $D \in \mathcal{X} \cap \mathcal{H} = \mathcal{F}$ and then $G \in \mathcal{X}$, a contradiction. Therefore $G \in \mathcal{H}$. Notice that $\mathrm{Soc}(G) \le O_p(G) = O_{p'p}(G)$ and so, $G/O_p(G) \in H(p)$. We have also by minimality that $G/O_p(G) \in \mathcal{X}$ and therefore $G/O_p(G) \in \mathcal{X} \cap \mathcal{H} = \mathcal{F}$. Hence $G/O_p(G) \in \mathcal{F} \cap H(p) = F(p)$ by (ii), and this implies that $G \in \mathcal{S}_p F(p) = F(p) \subseteq \mathcal{F}$ and then $G \in \mathcal{X}$, a contradiction.

If $\mathcal{F} \subseteq H(p)$ then by (i) $G \in \overline{X}(p) = \mathcal{H}_{F(p)} \subseteq \mathcal{H}_{\mathcal{F}} = \mathcal{X}$, a contradiction.

Thus, $\overline{\mathcal{X}} = \mathcal{X}$ and $\mathcal{X}$ is saturated. ∎

The main theorem and its consequences

The following propositions will be essential in the proof of the main theorem. Both are proved by similar arguments.

PROPOSITION 3: *If the formation $\mathcal{X} = \mathcal{H}_{\mathcal{F}}$ is saturated and we have* $\mathrm{char}(\mathcal{F}) = \mathrm{char}(\mathcal{H}) = \pi$ *then $\mathcal{F} \subseteq H(p)$ for every prime number $p \in \pi$.*

Proof: Suppose there exists a prime $p \in \pi$ such that $\mathcal{F} \not\subseteq H(p)$. By Lemma 1(ii) we have $X(p) = \mathcal{S}_p\mathcal{X}$ and by Proposition 2, $H(p) \cap \mathcal{F} = F(p)$.

Let G be a group of minimal order in $H(p) \setminus \mathcal{F}$; the group G is primitive and it can be factorized as $G = MN$ where N is a self-centralizing minimal normal q-subgroup of G, for some prime q, and M is a core-free maximal subgroup of G complementing N; clearly $M \in \mathcal{F}$ and $M \in H(q) \setminus F(q)$.

First notice that $q \neq p$ since otherwise we would have $G \in \mathcal{S}_p\mathcal{X} = \mathcal{X}$ and then $G \in \mathcal{H} \cap \mathcal{X} = \mathcal{F}$, a contradiction.

Denote by $\mathcal{I}$ a set composed by one and only one representative of each isomorphism class of irreducible $\mathrm{GF}(p)[M]$-modules. If $V \in \mathcal{I}$ denote by $P(V)$ its projective cover. Construct the direct sum $P = \oplus_{V\in\mathcal{I}}P(V)$. Since the regular module is faithful we have $1 = \mathrm{Ker}(M \,\mathrm{on}\, \mathrm{GF}(p)[M]) = \mathrm{Ker}(M \,\mathrm{on}\, P) = C_M(P)$ and P is faithful for M. Notice that $\mathrm{Soc}(P) = \oplus_{V\in\mathcal{I}}V$ and therefore it is the direct sum of pairwise non-isomorphic irreducible M-modules.

Construct the semidirect product $K = [P]M$. Since P is faithful, we have $\mathrm{Soc}(K) = \mathrm{Soc}(P)$. If V is a $\mathcal{H}$-central chief factor of K under P, then $M/C_M(V)$ is in $H(p)$ and therefore $M/C_M(V) \in H(p) \cap \mathcal{F} = F(p)$, i.e. V is $\mathcal{F}$-central. This means that $K \in \mathcal{X}$. Since $p \neq q$ we apply Cor. B.10.7 of [6] and there exists an irreducible and faithful K-module W over $\mathrm{GF}(q)$.

Construct the semidirect product $Y = [W]K$. Assume that W is $\mathcal{H}$-central in Y; then $K \in H(q) \cap \mathcal{X} \subseteq \mathcal{F}$. Since $q \in \pi$ we can say that $K/O_{q'}(K) \in F(q)$; but $O_{q'}(K) \leq C_K(P) = P$, that is $O_{q'}(K) = 1$, and $K \in F(q)$ and $M \in F(q)$, a contradiction. Hence W is $\mathcal{H}$-eccentric in Y and $Y \in \mathcal{X}$.

We can consider N as an irreducible K-module with $\mathrm{Ker}(K \,\mathrm{on}\, N) = P$; since W is a faithful K-module we can apply the Steinberg's theorem (see [6] Th. B.10.13) to conclude that there exists a natural number n such that N is an (irreducible) K-submodule of $W^{(n)} = \overbrace{W \otimes \cdots \otimes W}^{n \text{ times}}$.

Consider the Hartley group $H = H(W_1, \ldots, W_n)$, where $W_i \cong W$ for $i = 1, \ldots, n$ (see [6] pp. 197–203); then $H/\Phi(H) \cong W_1 \oplus \cdots \oplus W_n$ and $W^{(n)}$ is isomorphic to a subgroup in $Z(H) \cap \Phi(H)$. Construct the semidirect product $L = [H]K$. Then $L/\Phi(L) \in \mathrm{QR}_0(Y) \subseteq \mathcal{X}$; but in L there exists a minimal normal subgroup which is L-isomorphic to N and then it is $\mathcal{H}$-central and $\mathcal{F}$-eccentric; so $L \notin \mathcal{X}$. Therefore $\mathcal{X}$ is not a saturated formation, a contradiction.

Hence our claim is true: $\mathcal{F} \subseteq H(p)$ for every prime number $p \in \pi$. ∎

PROPOSITION 4: *If the formation $\mathcal{X} = \mathcal{H}_{\mathcal{F}}$ is saturated then*
(i) *$\mathcal{H}$ is of full characteristic, and*
(ii) *we have that* $\mathrm{char}(\mathcal{F}) = \mathrm{char}(\mathcal{X})$.

Proof: (i) The methods of the proof are analogous to those in Proposition 3.

Suppose that $\mathrm{char}(\mathcal{H}) = \pi$ is a proper subset of prime numbers and consider a prime number $p \in \pi'$.

Let G be a group of minimal order in $\mathcal{H} \setminus \mathcal{F}$; following the notation of Proposition 3 we can say that $q \neq p$ since $q \in \pi$. Clearly M is a p'-group. So, now $\mathrm{GF}(p)[M]$ is completely reducible and P is simply the direct sum $P = \bigoplus_{V \in \mathcal{I}} V$.

Construct the semidirect product $K = [P]M$. Notice that $K \notin \mathcal{H}$ since p divides $|K|$. Since P is faithful for M we have $\mathrm{Soc}(K) = P$. Clearly $K \in \mathcal{X}$ and since $p \neq q$ there exists an irreducible and faithful K-module W over $\mathrm{GF}(q)$.

Construct the semidirect product $Y = [W]K$. Since $Y/C_Y(W) \cong K \notin H(q)$ we have that W is $\mathcal{H}$-eccentric in Y and $Y \in \mathcal{X}$.

Arguing as in Proposition 3 we obtain that $\mathcal{X}$ is not a saturated formation; this is again a contradiction. So, $\mathcal{H}$ is of full characteristic.

(ii) If $p \in \mathrm{char}(\mathcal{F})$ then $C_p \in \mathcal{F} \subseteq \mathcal{X}$ and then $p \in \mathrm{char}(\mathcal{X})$. If $p \notin \mathrm{char}(\mathcal{F})$ then $C_p \in \mathcal{H} \setminus \mathcal{F}$ and then $C_p \notin \mathcal{X}$ and $p \notin \mathrm{char}(\mathcal{X})$. ∎

With this, if the formation $\mathcal{H}_{\mathcal{F}}$ is saturated we distinguish two possibilities:
(A) $\mathrm{char}(\mathcal{F}) = \pi$ is a proper subset of prime numbers, or
(B) $\mathcal{F}$ is of full characteristic.

Let us analyze first the case (A). Denote by $\mathcal{N}$ the class of all nilpotent groups.

LEMMA 3: *If $\mathcal{X} = \mathcal{H}_{\mathcal{F}}$ is saturated, $\mathcal{H}$ is of full characteristic and $\mathrm{char}(\mathcal{F}) = \pi$ is a proper subset of prime numbers, then $\mathcal{S}_p\mathcal{F} \subseteq \mathcal{H}$ for every prime number p. Hence, $\mathcal{N}\mathcal{F} \subseteq \mathcal{H}$.*

Proof: Suppose that $\mathcal{S}_p\mathcal{F} \nsubseteq \mathcal{H}$ and consider a group G of minimal order in $\mathcal{S}_p\mathcal{F} \setminus \mathcal{H}$; then G is a primitive group, $G = MN$ where $N = \mathrm{Soc}(G)$ is a selfcentralizing minimal normal p-subgroup of G and M is a core-free maximal subgroup and $M \in \mathcal{F}$. Since $G \notin \mathcal{H}$ it follows that $M \in \mathrm{Nor}_{\mathcal{H}}(G)$ and then $G \in \mathcal{X}$. Since p divides $|G|$ and $\mathcal{X}$ is saturated we have $p \in \mathrm{char}(\mathcal{X}) = \mathrm{char}(\mathcal{F}) = \pi$. Take a prime $q \in \pi'$ and consider an irreducible and faithful G-module V over $\mathrm{GF}(q)$; construct the semidirect product $Y = [V]G$. Now $Y \notin \mathcal{H}$ and therefore $\mathrm{Nor}_{\mathcal{H}}(G) \subseteq \mathrm{Nor}_{\mathcal{H}}(Y)$ and $Y \in \mathcal{X}$; since q divides $|Y|$ and $\mathcal{X}$ is saturated we have that $q \in \pi$, a contradiction. Hence $\mathcal{S}_p\mathcal{F} \subseteq \mathcal{H}$ for every prime number p.

Take now a group G of minimal order in $\mathcal{NF} \setminus \mathcal{H}$. Then $G = MN$ where $N = \mathrm{Soc}(G)$ is a selfcentralizing minimal normal p-subgroup of G and M is a core-free maximal subgroup and $M \in \mathcal{F}$. But this means that $G \in \mathcal{S}_p\mathcal{F} \subseteq \mathcal{H}$, a contradiction. Thus, $\mathcal{NF} \subseteq \mathcal{H}$. ∎

Let us examine now case (B).

LEMMA 4: *If the formation $\mathcal{H}_{\mathcal{F}}$ is saturated and $\mathcal{H}$ and $\mathcal{F}$ are both of full characteristic then $\mathcal{NF} \subseteq \mathcal{H}$.*

Proof: In this case $\mathrm{char}(\mathcal{F}) = \mathrm{char}(\mathcal{H})$ and we can apply Proposition 3 to obtain that for all primes p we have $\mathcal{F} \subseteq H(p)$. Then we have $\mathcal{NF} \subseteq \mathcal{H}$. ∎

Now we can state the main theorem.

THEOREM: *Suppose that $\mathcal{H}$ is a saturated formation and $\mathcal{F}$ is a proper saturated subformation of $\mathcal{H}$. Then the following statements are pairwise equivalent:*

(i) *the formation $\mathcal{H}_{\mathcal{F}}$ is saturated;*
(ii) $\mathcal{NF} \subseteq \mathcal{H}$;
(iii) $\mathcal{H}_{\mathcal{F}} = \mathcal{F}$.

Proof: (i)⇒(ii). If $\mathrm{char}(\mathcal{F})$ is a proper subset of prime numbers we apply Lemma 3 to deduce that $\mathcal{NF} \subseteq \mathcal{H}$. If $\mathcal{F}$ is of full characteristic the conclusion follows from Lemma 4.

(ii)⇒(iii). Suppose there exists a group G of minimal order in $\mathcal{H}_{\mathcal{F}} \setminus \mathcal{F}$; clearly $G \in \mathcal{H}_{\mathcal{F}} \cap \mathcal{NF}$, that is $G \in \mathcal{H}_{\mathcal{F}} \cap \mathcal{H} = \mathcal{F}$, a contradiction. Therefore $\mathcal{H}_{\mathcal{F}} = \mathcal{F}$.

(iii)⇒(i). It is trivial. ∎

Example 1: If $\mathcal{H} = \mathcal{S}_{p'}\mathcal{S}_p$, the class of p-nilpotent groups, then $\mathcal{H}_{\mathcal{F}}$ is saturated if and only if either

(i) $\mathcal{F} = (1)$, and then $\mathcal{H}_{\mathcal{F}} = (1)$, or

(ii) $\mathcal{F} = \mathcal{S}_p$, and then $\mathcal{H}_{\mathcal{F}} = \mathcal{S}_p$, or
(iii) $\mathcal{F} = \mathcal{H}$, and then $\mathcal{H}_{\mathcal{F}} = \mathcal{S}$.

Obviously when $\mathcal{F} = \mathcal{H}$ we have that $\mathcal{H}_{\mathcal{F}} = \mathcal{S}$ and if $\mathcal{F} = (1)$ then $\mathcal{H}_{\mathcal{F}} = (1)$. So, suppose $(1) \subset \mathcal{F} \subset \mathcal{H}$.

Let p be a prime in $\mathrm{char}(\mathcal{F})$; if there exists another prime $q \in \mathrm{char}(\mathcal{F})$, $q \neq p$, then the group $E(q/p) \in \mathcal{N}\mathcal{F} \setminus \mathcal{H}$ (where $E(q/p)$ is the group of [6] B.12.5) and therefore $\mathcal{H}_{\mathcal{F}}$ is not saturated by the above Theorem.

Therefore $\mathcal{F} = \mathcal{S}_p$. In this case we have indeed $\mathcal{N}\mathcal{S}_p \subseteq \mathcal{H}$ and by the Theorem we obtain the conclusion. ∎

Example 2: If $\mathcal{H} = \mathcal{U}$, the class of supersoluble groups, then $\mathcal{H}_{\mathcal{F}}$ is saturated if and only if $\mathcal{F} = (1)$ (and then $\mathcal{H}_{\mathcal{F}} = (1)$) or $\mathcal{F} = \mathcal{H}$ (and then $\mathcal{H}_{\mathcal{F}} = \mathcal{S}$).

We only have to show that if $\mathcal{F}$ is a saturated formation such that $\mathcal{N}\mathcal{F} \subseteq \mathcal{U}$ then $\mathcal{F} = (1)$.

Let p be a prime in $\mathrm{char}(\mathcal{F})$ and q another prime. Denote by P an extraespecial group of order p^3 and let V be an irreducible and faithful P-module over $\mathrm{GF}(q)$; construct the semidirect product $G = [V]P$. Since $G/C_G(V) = G/V \cong P$ we have $\dim V > 1$ and $G \notin \mathcal{U}$ althought $G \in \mathcal{N}\mathcal{F}$.

Therefore $\mathcal{H}_{\mathcal{F}}$ is not saturated. ∎

Our purpose now is to characterize the saturated formations $\mathcal{H}$ such that for every saturated formation $\mathcal{F}$, the class $\mathcal{H}_{\mathcal{F}}$ is a saturated formation. It is clear that for all saturated formations $\mathcal{F}$, we have $(1)_{\mathcal{F}} = \mathcal{S}$ and $\mathcal{S}_{\mathcal{F}} = \mathcal{F}$. What we prove next is that in fact $\mathcal{S}$ and (1) are the only saturated formations with this property.

COROLLARY: *Let $\mathcal{H}$ be a saturated formation; the following statements are equivalent:*

(i) *for every saturated formation $\mathcal{F}$, the formation $\mathcal{H}_{\mathcal{F}}$ is saturated;*
(ii) *$\mathcal{H}$ is either (1) or $\mathcal{S}$.*

Proof: Since $\mathcal{H}$ must be of full characteristic, arguing by induction, we have that the class $\mathcal{N}^k$ of all nilpotent groups of length at most k is contained in $\mathcal{H}$ for every k; so, in fact, $\mathcal{S} \subseteq \mathcal{H}$, and $\mathcal{H} = \mathcal{S}$. ∎

Final remark

In [4], N. Müller presents an approach to these questions from the point of view of the pronormal maximal subgroups. More precisely, he proves that if $\mathcal{H}$ is a saturated formation of full characteristic then the formation $\mathcal{X} = \mathcal{H}_{\mathcal{N}}$ is saturated if and only if $\mathcal{X}$ coincides with the class of all soluble groups whose maximal subgroups are $\mathcal{H}$-pronormal. Arguing similarly it is easy to give another characterization of the saturation of $\mathcal{H}_{\mathcal{F}}$ in our case (B):

THEOREM: *If $\mathcal{N} \subseteq \mathcal{F} \subseteq \mathcal{H}$ then $\mathcal{H}_{\mathcal{F}}$ is saturated if and only if $\mathcal{H}_{\mathcal{F}}$ is the class of all groups whose $\mathcal{F}$-pronormal maximal subgroups are $\mathcal{H}$-pronormal.*

ACKNOWLEDGEMENT: The research of the first and the third authors has been supported by *Proyecto PB 90-0414-C03-03* of DGICYT, Ministerio de Educación y Ciencia of Spain.

The first and second authors participate in the "*Acción Integrada Hispano-Alemana*" HA-010-91.

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